

Perturbed Kepler problem in general relativity with quaternions

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Abstract

The motion of binary star systems is re-examined in the presence of perturbations from the theory of general relativity. To handle the singularity of the Kepler problem, the equation of motion is regularized and linearized with quaternions. In this way first order perturbation results are derived using the quaternion based approach.

1 Introduction

In this paper gravitational effects as perturbations of the Kepler problem are examined with post-Newtonian approximation. Gravitational effects become strong when the components of the binary are close to each other, and the orbital separation is small.¹ The Kepler problem is singular when the separation is zero, therefore to study gravitational effects the desingularization – or regularization – of the equation of motion would be a substantial step.

It is well known that Kustaanheimo (1964) solved the regularization of the three-dimensional Kepler problem with spinors[1], which was reformulated by Stiefel[2]. In their method – the KS method for short – the regularization was carried out in four dimensions, and it was proved that the three-dimensional Kepler problem can only be regularized using four-dimensional linear spaces.

We follow another approach developed by J. Vrbik. In his work the mentioned four-dimensional space is the linear space of quaternions and the regularization is calculated with quaternion algebra. He applied his method with

¹The post-Newtonian approximation is not valid when the orbital separation is smaller than the innermost stable circular orbit or when the bodies start to merge. Therefore in the paper it is supposed that these limits are not reached.

success to the Lunar problem[3], and several perturbative forces were studied in details[4, 5, 6].

In the present work we use his method to examine gravitational effects analytically with quaternions. The leading order correction of general relativity to classical mechanics is calculated first. The formula for the precession of the pericentre is derived based on the Vrbik's quaternion formulae. Then the gravitational radiation reaction is analyzed, where the famous Peters-Mathews formula is proved[7]. In this calculation we manage to remove the residual coordinate gauge freedom of the gravitational reaction from the quaternionic equation of motion. In addition using a one-dimensional model we demonstrate that the regularization can lead to different results depending on that the Sundman transformation is employed with the perturbed or unperturbed orbital separation.

The regularization is defined with four-dimensional spaces, thus an additional geometrical constraint – a gauge – have to be applied to describe the three-dimensional spatial Kepler problem. In the KS method the so-called *bilinear relation* is defined, which is an excellent gauge for *numerical* calculations. Vrbik proposes another constraint to provide an *analytic* perturbative method, since – according to Vrbik – the bilinear relation is too restrictive to build an analytic perturbative method. This constraint is the major difference between the KS method and Vrbik's work.

The Laplace vector is a constant of motion of the Kepler problem, which is a consequence of the hidden symmetry of the problem[8]. This symmetry becomes manifest in four dimensions, which shows that the Kepler problem has another interesting connection with the four-dimensional space. This connection has far reaching consequences[9, 10].

Quaternions were first applied to regularize the Kepler problem by Chelnokov who successfully regularized the Kepler problem to describe rotating coordinate systems[11]. Moreover he was able to apply his results to describe the optimal control problem of a spacecraft[12].

Later it was shown by Vivarelli (1983) in a general mathematical sense that the KS method can be transformed identically into quaternion algebra[13]. Quaternion algebra proved to be very useful to derive the central ideas of the KS method. Remarkably the bilinear relation is described as a fibration of the quaternion space.

More recently Waldvogel showed that the spatial Kepler motion can be elegantly formulated with quaternions using a novel star conjugation operator[14]. The star conjugation is especially useful to handle the bilinear relation. The interesting connection with the Birkhoff transformation is also shown [15]. Quaternions turned to be useful in case of three and N-body applications[16].

It has to be emphasized that the mentioned quaternion approaches *exclusively* apply the “bilinear relation” as a gauge to reduce the dimensions from four to three, while Vrbik apply his special gauge.

This paper is organized as follows: a short outline of Vrbik's approach is provided in Section 2 and 3, where we describe the transformation of the Kepler problem into quaternion differential equation. Then the solution is given in

terms of ordinary differential equations of orbital elements. The advantages of Vrbik's calculus compared with the KS method are highlighted.

In Section 2.5 a one-dimensional example is given where we demonstrate that the result of the regularization depends on whether the Sundman transformation is applied with perturbed or unperturbed orbital separation.

In Section 4 Vrbik's method is applied to two perturbations. First of all, the leading order correction of general relativity to classical mechanics is examined. The formula for the precession of the pericentre is derived. Then the gravitational radiation reaction is analyzed, where the famous Peters-Mathews formula is proved using the quaternion approach [7]. In this calculation we solved to cancel the residual coordinate gauge freedom of the gravitational radiation reaction in the quaternionic equation of motion.

The conclusion and the outlook is given in Section 5 followed by the Appendix.

2 Linearization and regularization with quaternions

2.1 Quaternion algebra basics

The quaternion algebra has three imaginary units, generally called i , j and k , where $i^2 = j^2 = k^2 = -1$. Any of them anticommute

$$ij = -ji, jk = -kj, ki = -ik, \quad (1)$$

while the real unit 1 commutes with each of them. The four units together form the algebra's generators. Thus any element of the algebra can be written as²

$$\mathbb{A} = A + A_3i + A_2j + A_1k = A + \mathbf{a}. \quad (2)$$

Quaternion *conjugation* reverses the sign of the imaginary units

$$\bar{\mathbb{A}} = A - \mathbf{a}. \quad (3)$$

The *magnitude* of a quaternion is defined as

$$|\mathbb{A}| = \sqrt{\mathbb{A}\bar{\mathbb{A}}}. \quad (4)$$

Any quaternion can be written in the form $A + a\hat{\mathbf{a}}$, where a is the magnitude and $\hat{\mathbf{a}}$ is the unit direction of \mathbf{a} . Since $\hat{\mathbf{a}}^2 = -1$ the exponential on any quaternion can be expressed with Euler's formula

$$e^{A+a\hat{\mathbf{a}}} = e^A (\cos a + \hat{\mathbf{a}} \sin a). \quad (5)$$

²Note the unusual reversed ordering of the U_i components.

2.1.1 Representation of spatial vectors and rotations

Spatial vectors are represented with *pure* quaternions, which has no real part

$$\mathbf{x} = zi + yj + x\mathfrak{k}, \quad (6)$$

where the z -axis is associated with the i unit. With this interpretation quaternion multiplication can be expressed as

$$\mathbb{A}\mathbb{B} = (A + \mathbf{a})(B + \mathbf{b}) = AB - \mathbf{a} \cdot \mathbf{b} + A\mathbf{b} + B\mathbf{a} - \mathbf{a} \times \mathbf{b}. \quad (7)$$

By substituting $A = B = 0$ into this expression (7), the anticommutative cross product can be expressed as

$$\mathbf{a} \times \mathbf{b} = -\frac{\mathbf{a}\mathbf{b} - \mathbf{b}\mathbf{a}}{2}. \quad (8)$$

Let us introduce a vector \mathbf{w} . It is straightforward to show that a rotation around the vector \mathbf{w} with magnitude $|\mathbf{w}|$ can be written as[4]

$$\tilde{\mathbf{x}} = \bar{\mathbb{R}}\mathbf{x}\mathbb{R}, \quad (9)$$

where $\mathbb{R} = e^{\frac{\mathbf{w}}{2}}$. Note that $\mathbb{R}\bar{\mathbb{R}} = 1$, hence $\mathbf{x} = \mathbb{R}\tilde{\mathbf{x}}\bar{\mathbb{R}}$ is the inverse rotation. Let us suppose that the rotation is parameter dependent $\mathbb{R}(s)$, and differentiate it with respect to s

$$\tilde{\mathbf{x}}' = \bar{\mathbb{R}}\mathbf{x}\mathbb{R}' + \bar{\mathbb{R}}'\mathbf{x}\mathbb{R} = \tilde{\mathbf{x}}\bar{\mathbb{R}}\mathbb{R}' + \bar{\mathbb{R}}'\mathbb{R}\tilde{\mathbf{x}}, \quad (10)$$

where definition (9) of $\tilde{\mathbf{x}}$ was applied. With the help of the identity $(\bar{\mathbb{R}}\mathbb{R})' = \bar{\mathbb{R}}'\mathbb{R} + \bar{\mathbb{R}}\mathbb{R}' = 0$ and the cross product (8) this can be further written

$$\tilde{\mathbf{x}}' = \tilde{\mathbf{x}}\bar{\mathbb{R}}\mathbb{R}' - \bar{\mathbb{R}}'\mathbb{R}\tilde{\mathbf{x}} = \mathbb{Z} \times \tilde{\mathbf{x}}, \quad (11)$$

where $\mathbb{Z} = 2\bar{\mathbb{R}}\mathbb{R}'$. Consequently \mathbb{Z} is the instantaneous angular velocity of $\tilde{\mathbf{x}}$ with respect to s . With an inverse rotation the angular velocity \mathbb{Z} can be expressed in a special coordinate system – in the *Kepler frame* – where $\tilde{\mathbf{x}}$ is instantaneously at rest

$$\mathbb{Z}_K = \mathbb{R}\mathbb{Z}\bar{\mathbb{R}} = 2\mathbb{R}'\bar{\mathbb{R}}. \quad (12)$$

where the subscript indicates the Kepler frame.

2.2 The Kepler problem with quaternions

Equipped with the quaternion formulae we turn to regularize the Kepler problem with quaternions. The perturbed Kepler problem in the $G = c = 1$ system is given by the equation

$$\ddot{\mathbf{r}} + \frac{m}{r^3}\mathbf{r} = \varepsilon\mathbf{f}, \quad (13)$$

where \mathbf{r} is the orbital separation vector of the orbiting bodies

$$\mathbf{r} = \mathbf{r}_1 - \mathbf{r}_2, \quad r = |\mathbf{r}|, \quad (14)$$

ε is the small parameter of the perturbation, and $m = m_1 + m_2$ is the sum of the two masses.

To regularize the Kepler problem the separation is defined by the following *quaternionic* equation

$$\mathbf{r} = \bar{\mathbb{U}}\mathfrak{k}\mathbb{U}, \quad (15)$$

where $\mathbb{U} = U + U_3\mathbf{i} + U_2\mathbf{j} + U_1\mathbf{k}$ is a general quaternion. The conjugate of \mathbf{r} is

$$\bar{\mathbf{r}} = \overline{\bar{\mathbb{U}}\mathfrak{k}\mathbb{U}} = \bar{\mathbb{U}}\bar{\mathfrak{k}}\bar{\mathbb{U}} = -\bar{\mathbb{U}}\mathfrak{k}\mathbb{U} = -\mathbf{r}, \quad (16)$$

where the $\overline{\bar{\mathbb{A}}\mathbb{B}} = \bar{\mathbb{B}}\bar{\mathbb{A}}$, $\bar{\bar{\mathbb{A}}} = \mathbb{A}$ and $\bar{\mathfrak{k}} = -\mathfrak{k}$ properties were used. Therefore \mathbf{r} has no real part and can be written as $\mathbf{r} = zi + yj + xk$. A direct calculation from (15) tells us that

$$\begin{aligned} x &= U^2 + U_1^2 - U_2^2 - U_3^2, \\ y &= 2(U_1U_2 + U_3U), \\ z &= 2(U_1U_3 - U_2U), \end{aligned} \quad (17)$$

and the real part $UU_1 - U_3U_2 + U_2U_3 - U_1U = 0$ indeed vanishes. The obtained transformation (17) is just the KS transformation with the $U \rightarrow -U$ convention. [2]

Transformation (15) maps the four-dimensional quaternion space into the three-dimensional space of spatial vectors. Therefore the solution to a given three-dimensional \mathbf{r} in terms of four-dimensional \mathbb{U} is not unique. From (15) it is clear that the transformation

$$\mathbb{U} \rightarrow e^{\mathfrak{k}\alpha}\mathbb{U}, \quad (18)$$

where α is an arbitrary real number, is a continuous symmetry of (15). It follows that we have a one-dimensional compact manifold – a fibre – of \mathbb{U} s for a given \mathbf{r} , and (18) defines a fibration of the space of \mathbb{U} s. The geometrical background of this transformation is elegantly described in Waldvogel (2005) [14]. This additional degree of freedom will be constrained in a careful manner.

To complete the regularization the time has to be also transformed. The *Sundman transformation* is given by

$$\frac{dt}{ds} = 2r\sqrt{\frac{a}{m}}, \quad (19)$$

where s is the modified time and a is a real and at this point *arbitrary* function of s (it will be chosen such that it simplifies the solution). From now the operator $'$ indicates differentiation with respect to the modified time s .

Inserting the definition of the orbital separation (15) into the equation of motion (13) while transforming the original time into the modified one using (19) lead us to the following quaternionic differential equation [4]

$$2\mathbb{U}'' - (2\mathbb{U}'\bar{\mathbb{U}}' - 4a)\frac{\mathbb{U}}{r} + 2\mathfrak{k}\mathbb{U}'\frac{\Gamma}{r} + \mathfrak{k}\mathbb{U}\left(\frac{\Gamma}{r}\right)' - \left(\mathbb{U}' + \mathfrak{k}\mathbb{U}\frac{\Gamma}{2r}\right)\frac{a'}{a} + 4\frac{a}{m}\varepsilon\mathbb{U}\mathbf{r}\mathbf{f} = 0, \quad (20)$$

where

$$\Gamma = \overline{\mathbb{U}}\mathfrak{k}\mathbb{U}' - \overline{\mathbb{U}}'\mathfrak{k}\mathbb{U} = 2(U_1U' - UU'_1 + U_2U'_3 - U_3U'_2), \quad (21)$$

which is a scalar in the sense that it is invariant under conjugation. This quantity is the four-dimensional scalar product of the tangent vector of the $\mathbb{U}(s)$ curve and the tangent of the fibre at that point multiplied by two³. Let us define a condition

$$\Gamma = 0, \quad (22)$$

which means that the trajectory $\mathbb{U}(s)$ intersects the fibres under right angles. It is indeed a condition as transforming \mathbb{U} with symmetry transformation (18) Γ transforms as

$$\Gamma \rightarrow \Gamma - 2\alpha'r, \quad (23)$$

hence condition (22) can be satisfied by solving a first order differential equation for $\alpha(s)$.

The geometrical constraint (22) is equivalent with the so-called “*bilinear relation*”, which plays an essential role in the KS method[2]. It can be proved that (18) is a dynamical symmetry, since the transformed \mathbb{U} solves the equation of motion. Furthermore if condition (22) is satisfied then Γ' also vanishes, which means that one can maintain this condition by finding the proper initial conditions $\mathbb{U}(0)$ and $\mathbb{U}'(0)$ which satisfy the “bilinear relation” (22)[4].

2.3 Solving the unperturbed case: Kepler orbits

The unperturbed situation with condition (22) reduces the perturbed equation of motion (20) to the following equation

$$\mathbb{U}'' - (\mathbb{U}'\overline{\mathbb{U}}' - 2a)\frac{\mathbb{U}}{r} = 0. \quad (24)$$

The coefficient of \mathbb{U} is

$$\frac{\mathbb{U}'\overline{\mathbb{U}}' - 2a}{r} = \frac{2a}{m} \left(\frac{v^2}{2} - \frac{m}{r} \right) = \frac{2ah}{m}, \quad (25)$$

where h is a constant of motion. Let us fix the parameter a

$$a = -\frac{m}{2h}, \quad (26)$$

which means that a is the semimajor axis of the elliptical motion. With this choice the equation of motion is the harmonic oscillator with *constant* frequency

$$\mathbb{U}''_0 + \mathbb{U}_0 = 0, \quad (27)$$

where the subscript 0 indicates the unperturbed case.

³ This statement can be checked by differentiating the transformed \mathbb{U} in (18) with respect to α which produces the tangent of the fiber and then taking the four-dimensional scalar product with \mathbb{U}' .

The general solution of this second order differential quaternion equation has six free parameters as it is constrained with (22) and has a redundant phase (18). The trial solution of the unperturbed case can be parametrized as follows

$$\mathbb{U}_0 = a^{1/2} \beta_+^{-1/2} (q + \beta q^{-1}) \mathbb{R}, \quad (28)$$

where $\beta_{\pm} = 1 \pm \beta^2$ and $q = e^{\frac{i\omega}{2}}$ with $\omega = 2(s - s_p)$. In the next paragraph it is shown that the trial solution describes an elliptical Kepler orbit.

Let us set $\mathbb{R} = 1$ for the moment and substitute \mathbb{U}_0 into the definition of the separation (15)

$$\mathbf{r}_0 = a\beta_+^{-1} \mathfrak{k} (z + \beta^2 z^{-1} + 2\beta), \quad (29)$$

where $z = q^2$ and the identity $q\mathfrak{k} = \mathfrak{k}q^{-1}$ was applied. This formula can be further expanded using the $z = \cos \omega + i \sin \omega$ identity

$$\mathbf{r}_0 = a\mathfrak{k}(\cos \omega + 2\beta\beta_+^{-1}) + a\mathfrak{j}\beta_-\beta_+^{-1} \sin \omega, \quad (30)$$

which means that according to (6) \mathbf{r}_0 describes the following parametric curve

$$\begin{aligned} x_0 &= a(\cos \omega + 2\beta\beta_+^{-1}) = a(\cos \omega + e), \\ y_0 &= a\beta_-\beta_+^{-1} \sin \omega = a\sqrt{1 - e^2} \sin \omega, \end{aligned} \quad (31)$$

with $e = 2\beta\beta_+^{-1}$. These equations describe an ellipse in the (x, y) coordinate-plane, with semimajor axis a , and eccentricity e . From equations (31) it follows that $\omega = 0$ parametrizes the apocenter, thus ω is equivalent with the eccentric anomaly, except that the latter is zero at the pericenter. This tells us that s_p is the time advance of apocenter passage measured in modified time. In the general case \mathbb{R} is obviously the rotation between the orbital and reference frames, where the rotation according to (9) can be given with Euler angles

$$\mathbb{R} = e^{i\frac{\psi}{2}} e^{\mathfrak{k}\frac{\theta}{2}} e^{i\frac{\phi}{2}}. \quad (32)$$

The formulae which provide the connection between the a, β, s_p and angular parameters – the orbital elements – and the quaternion components are collected in the Appendix.

Despite of the great advantages of the gauge condition (22) - which is especially fine for numerical calculations - for perturbative calculations another geometrical condition is proposed.

2.4 The perturbed case

The trial solution for the perturbed equation of motion (20), is just the unperturbed solution form (28) completed with general ε proportional terms [4]

$$\mathbb{U} = a^{1/2} \beta_+^{-1/2} \left(q + \beta q^{-1} + q\mathbb{D} + \mathfrak{k}q \frac{i\mathbf{b} + \mathbb{S}}{1 + \beta z} \right) \mathbb{R}, \quad (33)$$

where both the \mathbb{D} and \mathbb{S} quaternions are $\mathcal{O}(\varepsilon)$ quantities, and complex in the sense that they are in the subspace spanned by the units 1 and \mathbf{i} , while the b quantity is real.

Using the definition of the separation (15) the perturbed separation vector in the Kepler frame is

$$\mathbf{r}_K = \mathbf{r}_{K,0} + a\beta_+^{-1} \{2\mathfrak{k}(z + \beta)\mathbb{D} - 2\mathbf{i}\text{Im}\mathbb{S} - 2\mathbf{i}b\}, \quad (34)$$

where the ε^2 terms were neglected. The result tells us that parameter b describes a translation along the \mathbf{i} unit which is a translation along the z -axis of the Kepler frame according to (6). By considering (30) it parametrizes a translation perpendicular to the orbital plane. The same is true for the imaginary part of \mathbb{S} , while the real part of \mathbb{S} has no physical effect. Parameter \mathbb{D} is complex and it is multiplied with \mathfrak{k} , thus the result is in the subspace spanned by the units \mathbf{j} and \mathfrak{k} . These units are associated (6) with the (x, y) coordinate plane of the Kepler frame, which is the orbital plane according to (30).

After introducing the trial solution in the perturbed case we fix the gauge. Vrbik's condition is that the real part of \mathbb{S} must vanish[4]

$$\mathbb{S}^* = -\mathbb{S}, \quad (35)$$

where the operator $*$ conjugates its complex quaternion argument. In this case the trial solution (33) has no \mathfrak{k} proportional part.

Transformation (18) has a simple geometrical interpretation. It describes a double rotation, one in the $(1, \mathfrak{k})$ and another one in the (\mathbf{i}, \mathbf{j}) subspace. Hence a transformation (18) on \mathbb{U} (33) whose tangent is the coefficient of the \mathfrak{k} part of the trial solution (33) divided by its real part cancels the coefficient of \mathfrak{k} . In the leading ε order this rotation is

$$\alpha = -\frac{\mathbb{S}^* + \mathbb{S}}{2(1 + \beta z^{-1})(1 + \beta z)}, \quad (36)$$

which can be fulfilled without solving any differential equation for α in contrary to (23).

2.5 Example for the regularization

The one-dimensional two-body problem is considered with the following special force $f = \varepsilon \dot{x}^2$, therefore

$$\ddot{x} + \frac{m}{x^2} = \varepsilon \dot{x}^2. \quad (37)$$

We have chosen this kind of special force since the equation of motion has a constant of motion⁴.

⁴ The first integral of the Eq. (37) is $y'^2 = Cy^2 e^{2\varepsilon y^2} / 4 + m/2 + \varepsilon m y^2 e^{2\varepsilon y^2} Ei(-2\varepsilon y^2)$, where C is the arbitrary constant of motion ($C = 2E_0$ for unperturbed motion) and $Ei(x) = -\int_{-x}^{\infty} t^{-1} e^{-t} dt$ is the exponential integral function.

Eq. (37) can be regularized with the following transformations

$$x = y^2, \quad (38)$$

$$\frac{dt}{ds} = x. \quad (39)$$

Then Eq. (37) is

$$y'' + \frac{m - 2(y')^2}{2y} = 2\varepsilon y(y')^2. \quad (40)$$

In case of unperturbed motion ($\varepsilon = 0$) the energy is the constant of motion ($E_0 = 2(y')^2/y^2 - m/y^2$) and Eq. (40) is the equation of the harmonic oscillator

$$y_0'' - \frac{E_0}{2} y_0 = 0, \quad (41)$$

where clearly for bounded motion $E_0 < 0$. The general solution for Eq. (41) is $y_0 = C_1 e^{i\Omega s} + C_2 e^{-i\Omega s}$, where $\Omega = \sqrt{|E_0|}/2$ is the orbital frequency.

We assume that for perturbed motion ($\varepsilon \neq 0$) the form of the solution is $y_\varepsilon = y_0 + \varepsilon \delta$. Then Eq. (40) in the leading order of δ is

$$\delta'' - \frac{2y_0'}{y_0} \delta' - \frac{E_0}{2} \delta = 2y_0(y_0')^2, \quad (42)$$

and using the $y_0 = A \cos(\Omega s)$ solution of the unperturbed motion in (42) we get

$$\delta'' + 2\Omega \tan(\Omega s) \delta' + \Omega^2 \delta = 2\Omega^2 A^3 \cos(\Omega s) \sin^2(\Omega s), \quad (43)$$

where the sign of $\Omega^2 \delta$ is positive, since $E_0 < 0$. Numerical solutions for Eq. (43) can be seen on Fig. 1. It can be seen that the numerical solution $\delta(s)$ of these two examples are well-behaving, bounded functions for various initial values.

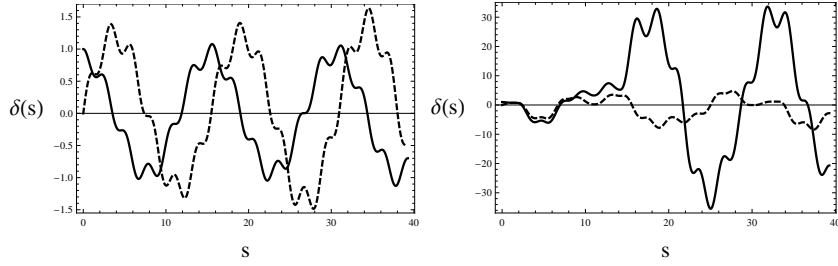


Figure 1: The homogeneous (left) and inhomogeneous (right) solutions for Eq. (43) for $A = 1 = \Omega$, $\delta'(0) = 0, \delta(0) = 1$ (dashed line) or $\delta'(0) = 1, \delta(0) = 0$ (line).

So far we have represented the regularization of the one-dimensional perturbed two-body problem with a *heuristic* special force. Let us consider the one-dimensional model using the *generalized* Sundman transformation

$$\frac{dt}{ds} = \tilde{x}, \quad (44)$$

where \tilde{x} is not fixed yet. Eq. (37) can be regularized using transformations (38) and (44)

$$y'' + \frac{(y')^2}{y} - \frac{\tilde{x}'y'}{\tilde{x}} + \frac{m\tilde{x}^2}{2y^5} = 2\varepsilon y (y')^2. \quad (45)$$

If $\tilde{x} = x$ (*desingularized* in perturbed orbit) we obtain Eq. (42). If $\tilde{x} = x_0$ (*desingularized* in unperturbed orbit) the result is

$$y'' + \frac{(y')^2}{y} - \frac{x'_0 y'}{x_0} + \frac{mx_0^2}{2y^5} = 2\varepsilon y (y')^2. \quad (46)$$

Substituting the $y_\varepsilon = y_0 + \varepsilon\delta_0$ perturbed solution, one obtains

$$\delta_0'' - \frac{E(y_0)}{2}\delta_0 = 2\varepsilon y_0 (y_0')^2, \quad (47)$$

where $E(y_0) = [2(y_0')^2 + 5m]/y_0^2$ is not the constant of motion (note that the coefficient of the linear term is the constant of motion in case $\tilde{x} = x$ (Eq. (42)).

In total two types of desingularization ($\tilde{x} = x, x_0$) using the $y_0 = A \cos(\Omega s)$ (we have fixed the frequency $\Omega = 1$) unperturbed solution can be given

$$\delta'' + 2 \tan(s)\delta' + \delta = 2A^3 \cos s \sin^2 s, \quad (48)$$

$$\delta_0'' - \left(\frac{5m}{2A^2} \sec^2 s + \tan^2 s \right) \delta_0 = 2A^3 \cos s \sin^2 s. \quad (49)$$

The numerical analyzis of these two equations with different initial values can be seen on Fig. 2.5.

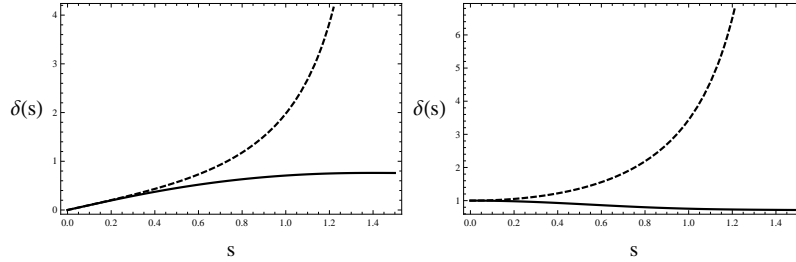


Figure 2: The numerical solutions for (48) and (49) with different initial values

It can be seen that in this one-dimensional perturbed two-body problem the two types of desingularization methods lead to quite different solutions. Therefore the Sundman transformation is generally nontrivial in perturbed equations.

3 Orbital elements with quaternions

Before explaining the quaternion approach the classical equations are described in order to explain the relationship between the two different methods. The

equation system of the classical two-body problem is of total order six, hence it can be described with six first integrals, which are also called *orbital elements*. These elements are the *semi-major axis* a , the *eccentricity* e , the *inclination* θ , *longitude of the ascending node* ϕ , the *argument of the pericenter* ψ ⁵ and the *mean anomaly at the epoch* l_0 (or time of pericenter passage t_0) related to the dynamics. The *Lagrange planetary equations* in the standard perturbed two-body problem are[17]

$$\begin{aligned}
\frac{da}{dt} &= \frac{2}{n\sqrt{1-e^2}} \left(Se \sin \chi + T \frac{a(1-e^2)}{r} \right), \\
\frac{de}{dt} &= \frac{\sqrt{1-e^2}}{na} [S \sin \chi + T (\cos \chi + \cos \xi)], \\
\frac{d\theta}{dt} &= \frac{r \cos(\chi + \psi)}{na^2 \sqrt{1-e^2}} W, \\
\frac{d\phi}{dt} &= \frac{r \sin(\chi + \psi)}{na^2 \sqrt{1-e^2} \sin \theta} W, \\
\frac{d\psi}{dt} &= -\cos \theta \frac{d\phi}{dt} + \frac{\sqrt{1-e^2}}{nae} \left[T \left(1 + \frac{r}{a(1-e^2)} \right) \sin \chi - S \cos \chi \right], \\
\frac{dl_0}{dt} &= -\sqrt{1-e^2} \left(\frac{d\psi}{dt} + \cos \theta \frac{d\phi}{dt} \right) - S \frac{2r}{na^2},
\end{aligned} \tag{50}$$

where χ is the *true anomaly*, ξ is the *eccentric anomaly* and r is the parametrization of the osculating orbit

$$r = \frac{a(1-e^2)}{1+e \cos \chi} = a(1-e \cos \xi), \tag{51}$$

and l is the *mean anomaly*, which can be defined by the Kepler equation

$$l - l_0 = n(t - t_0) = \xi - e \sin \xi, \tag{52}$$

and $n = m^{1/2} a^{-3/2}$ is the *mean motion*.

The S, T quantities are the projections of the perturbing force to the orbital plane, while W is the projection to the normal vector of the orbital plane $\hat{\mathbf{k}}$

$$S = \hat{\mathbf{r}} \cdot \mathbf{f}, \quad T = (\hat{\mathbf{k}} \times \hat{\mathbf{r}}) \cdot \mathbf{f}, \quad W = \hat{\mathbf{k}} \cdot \mathbf{f}. \tag{53}$$

To derive *quaternion* differential equations for the orbital elements the trial solution (33) has to be substituted into the equation of motion (20). To simplify the calculation the quaternion equation of motion (20) is decoupled into two complex equations. The derivation of the complex equations are given in details[4, 18] and the most important steps are briefly outlined in our Appendix. Here only the solution and the necessary definitions are presented.

⁵in classical celestial mechanics the symbols are ι , Ω and ω respectively. Here we adopted the notations of J.Vrbik.

The following auxiliary quaternion quantities have to be defined

$$\begin{aligned}\mathbb{Q} &= -2\varepsilon \frac{a}{m} \frac{Cx(\mathbf{r}_K \mathbf{f}_K)}{1 + \beta z} = -2\varepsilon \frac{a}{m} \frac{Cx(\mathbf{r}_{K0} \mathbf{f}_{K0})}{1 + \beta z} + O(\varepsilon^2), \\ \mathbb{W} &= -4\varepsilon \frac{a}{m} r Cx(\mathbf{f}_K) = -4\varepsilon \frac{a}{m} r_0 Cx(\mathbf{f}_{K0}) + O(\varepsilon^2),\end{aligned}\quad (54)$$

where the operator Cx is a projector, which projects its quaternion argument to the complex subspace spanned by the units 1 and \mathbf{i} . The additional subscript 0 indicates the unperturbed value of the symbol. The subscript K is omitted in r and r_0 as they are scalars and have the same value in every frame. To point out the relationship between the quaternion formulae and the classical equations (50) note that the real and imaginary part of the complex \mathbb{Q} quantity is proportional to the previously introduced S and T (53) respectively, while \mathbb{W} is proportional to W .

The quaternion coefficients (54) can be expanded into Laurent series[4]

$$\mathbb{Q} = \sum_{n=-\infty}^{n=+\infty} \mathcal{Q}_n z^n, \quad \mathbb{W} = \sum_{n=-\infty}^{n=+\infty} \mathcal{W}_n z^n, \quad (55)$$

together with \mathbb{D} and \mathbb{S} from (33)

$$\mathbb{D} = \sum_{\substack{n=-\infty \\ n \neq -1, 0}}^{n=+\infty} \mathcal{D}_n z^n, \quad \mathbb{S} = \sum_{n=2}^{n=+\infty} \mathcal{S}_n z^n. \quad (56)$$

The Laurent series are given in powers of z . From the definition of the orbital separation \mathbf{r} (15) follows that this is enough as the expansion of the separation contains every power of q . The coefficients \mathcal{D}_{-1} , \mathcal{D}_0 and \mathcal{S}_{-1} , \mathcal{S}_1 were left out from the expansion of \mathbb{D} and \mathbb{S} since they would only duplicate the q and q^{-1} terms of the solution (33), while \mathcal{S}_0 was explicitly separated as b .

Substituting expansions (55) and (56) into the complex equations (92) and (93) the differential equations for the orbital elements can be extracted by matching the coefficients of z with the same power on both side of the equation.

The obtained differential equations are the following[4, 18]

$$a' = 2a \operatorname{Im}(\mathcal{Q}_0 - \beta \mathcal{Q}_{-1}), \quad (57)$$

$$\beta' = -\frac{\beta_+}{4} \operatorname{Im}(\mathcal{Q}_1 + 3\beta \mathcal{Q}_0 + 3\mathcal{Q}_{-1} + \beta \mathcal{Q}_{-2}), \quad (58)$$

$$Z_1 = -\beta_-^{-1} \operatorname{Im}\left(\frac{\beta_+}{2} \mathcal{W}_1 + \beta \mathcal{W}_0\right), \quad (59)$$

$$Z_2 = -\frac{1}{2} \operatorname{Re}(\mathcal{W}_1), \quad (60)$$

$$Z_3 = \frac{1}{4\beta} \operatorname{Re}\{-\beta_+ \mathcal{Q}_1 + \beta(1 - 3\beta^2) \mathcal{Q}_0 + (3 - \beta^2) \mathcal{Q}_{-1} + \beta \beta_+ \mathcal{Q}_{-2}\}, \quad (61)$$

$$s'_p = \frac{Z_3}{2} + \frac{\beta_+^{-1}}{4} \operatorname{Re}\{\beta(2 + \beta^2) \mathcal{Q}_1 + (\beta_+ + 3\beta^4) \mathcal{Q}_0 - \beta(1 - 2\beta^2) \mathcal{Q}_{-1} - \beta^4 \mathcal{Q}_{-2}\}, \quad (62)$$

$$b = \frac{1}{8} \operatorname{Im}\{(\beta_-^2 \mathcal{W}_0 + 2\beta^2 \mathcal{W}_2) \beta_+^{-1} + \beta \mathcal{W}_1\}, \quad (63)$$

and the two additional formulae for \mathbb{D} and \mathbb{S} is given in the Appendix. The Z_i quantities are the components of \mathbb{Z}_K , where the Kepler frame subscript was dropped to simplify the notation. Note that equation (61) is singular in β , which shows that in the circular orbit limit the ordinary sense of the rotation no longer valid.

The coefficients \mathcal{Q}_n or \mathcal{W}_n of the Laurent series can be obtained with a contour integral, where C_0 is the unit circle

$$\mathcal{Q}_n = \oint_{C_0} \frac{\mathbb{Q}}{z^n} \frac{dz}{2\pi i z}. \quad (64)$$

Note that the Laurent expansion (56) of \mathbb{D} and \mathbb{S} has simplified the form of the differential equations (57)-(63) with respect to the Lagrange's planetary equations (50). The Laurent series of \mathbb{D} and \mathbb{S} absorbed the “short” term oscillatory part of the equation. The remaining differential equations contain only the adiabatic, “long” term part, which might be easier to solve.

We have to amend the equations above with the transformation of the angular velocity from the comoving Kepler frame to the inertial system, which are the following

$$\begin{aligned} \phi' &= \frac{Z_1 \sin \psi + Z_2 \cos \psi}{\sin \theta}, \\ \theta' &= Z_1 \cos \psi - Z_2 \sin \psi, \\ \psi' &= Z_3 - \phi' \cos \theta. \end{aligned} \quad (65)$$

This transformation is familiar from classical mechanics, in deriving the Euler equations of the rigid body.

4 GR perturbations

In this section perturbations calculated from the general relativity are examined using the described quaternion approach. The perturbations are examined with post-Newtonian approximation. The post-Newtonian approximation applies an expansion of corrections to the Newtonian gravitational theory with an expansion parameter $\varepsilon \approx v^2 \approx m/r$, which is supposed to be small, where v is the velocity.

We use equations up to $\varepsilon^{5/2}$, (post) $^{5/2}$ -Newtonian order, which is the order where the dominant gravitational radiation damping forces occur.

First of all, the (post) 1 -Newtonian correction to the classical mechanics will be examined in the first section. This is followed by the (post) $^{5/2}$ -Newtonian analysis of gravitational radiation where we rederive the classical Peters-Mathews formula.

4.1 Planar assumption

The mentioned perturbations are *planar* perturbations, in the sense that the force lies within the orbital plane. In this case obviously $\mathbb{S} = b = 0$ and the trial solution (33) contains only perturbations within the orbital plane

$$\mathbb{U}_K = a^{1/2} \beta_+^{-1/2} (q + \beta q^{-1} + q\mathbb{D}) , \quad (66)$$

It follows that in case of planar forces the $\mathbb{S}^* = -\mathbb{S}$ condition (35) is true. Remarkably the $\Gamma = 0$ condition is also satisfied[4]. To show this we need the derivative of \mathbb{U} expressed with Kepler frame quantities

$$\mathbb{U}' = \mathbb{U}'_K \mathbb{R} + \mathbb{U}_K \mathbb{R}' = \left(\mathbb{U}'_K + \mathbb{U}_K \frac{\mathbb{Z}_K}{2} \right) \mathbb{R}. \quad (67)$$

Therefore

$$\Gamma = 2 \operatorname{Re} (\bar{\mathbb{U}} \mathfrak{k} \mathbb{U}') = 2 \operatorname{Re} \left\{ \bar{\mathbb{R}} \bar{\mathbb{U}}_K \mathfrak{k} \left(\mathbb{U}'_K + \mathbb{U}_K \frac{\mathbb{Z}_K}{2} \right) \mathbb{R} \right\} , \quad (68)$$

and since the rotation can be dropped under the real part operator

$$\Gamma = 2 \operatorname{Re} \left(\bar{\mathbb{U}}_K \mathfrak{k} \mathbb{U}'_K + \mathbf{r}_K \frac{\mathbb{Z}_K}{2} \right) . \quad (69)$$

In the planar case \mathbb{U}_K is a complex number, therefore both $\bar{\mathbb{U}}_K \mathfrak{k} \mathbb{U}'_K$ and \mathbf{r}_K are in the orbital plane spanned by the \mathbf{j} and \mathfrak{k} units. In the planar case the orbital plane is preserved, therefore \mathbb{Z}_K must be perpendicular to this \mathbf{j} , \mathfrak{k} subspace. It means that \mathbb{Z}_K has only \mathbf{i} part. It is easy to see from equation (69) that the argument of the operator Re has no real part. Therefore in case of planar forces Γ vanishes.

Consequently the equation of motion (20) simplifies

$$2\mathbb{U}'' - (2\mathbb{U}'\bar{\mathbb{U}}' - 4a) \frac{\mathbb{U}}{r} - \mathbb{U}' \frac{a'}{a} + 4 \frac{a}{m} \varepsilon \mathbb{U} \mathbf{r} \mathbf{f} = 0 . \quad (70)$$

Let us introduce a τ parameter by rescaling the modified time $d\tau = 2a^{1/2}m^{-1/2}ds$. With the help of the τ parameter the equation of motion is just the perturbed harmonic oscillator

$$2\frac{d^2\mathbb{U}}{d\tau^2} - h\mathbb{U} + \varepsilon\mathbb{U}\mathbf{r}\mathbf{f} = 0. \quad (71)$$

In the planar case the equation of motion substantially simplified and identical with the equation of Waldvogel[14].⁶

In the planar case the perturbations \mathbb{D} can be expressed in a more conventional way. Let us introduce $a = a_0 + \delta a$ and $\beta = \beta_0 + \delta\beta$ in the trial solution (28) where δa and $\delta\beta$ are first order quantities. In this case by matching the first order part of Eqs. (28) and (66) one obtains the following important relations

$$\delta a = 2a_0 \frac{\text{Im}(B\mathbb{D})}{\text{Im}(A^*B)}, \quad \delta\beta = 2 \frac{\text{Im}(A\mathbb{D})}{z^{-1} - z}, \quad (72)$$

where $A = 1 + \beta_0 z$ and $B = z - \beta_0(1 - \beta_0)^2 A$.

4.2 The classical post-Newtonian effect

In this section the leading contribution of general relativity to classical Newtonian mechanics is examined in details. The force is given by numerous authors [19]

$$\mathbf{a}_{PN} = -\frac{m}{r^2} \left\{ \hat{\mathbf{n}} \left[(1 + 3\eta) v^2 - 2(2 + \eta) \frac{m}{r} - \frac{3}{2} \eta \dot{r}^2 \right] - 2(2 - \eta) \dot{r} \mathbf{v} \right\}. \quad (73)$$

where the subscript PN denotes the post-Newtonian term, $\hat{\mathbf{n}} = \mathbf{r}/r$, $\eta = (m_1 m_2)/m^2$ and $v = |\mathbf{v}|$ is the absolute value of the orbital velocity $\mathbf{v} = d\mathbf{r}/dt$.

After transforming it to quaternion expression

$$\begin{aligned} \mathbf{a}_{PN} = & -\frac{\mathbf{r}_K}{r^5} \frac{m}{4a} (1 + 3\eta) (\mathbf{r}'_K \mathbf{r}'_K) + \frac{\mathbf{r}_K}{r^4} 2(2 + \eta) m^2 \\ & + \frac{\mathbf{r}_K}{r^5} \frac{3}{8} \frac{m^2 \eta}{a} (r')^2 + \frac{\mathbf{r}'_K}{r^4} \frac{m^2}{2a} (2 - \eta) r'. \end{aligned} \quad (74)$$

This result (74) have to be substituted into the definition of \mathbb{Q} (54), where the separation \mathbf{r}_K , its magnitude r and their derivatives are treated as functions of z according to (29). Therefore the result is a function of z

$$\begin{aligned} \mathbb{Q} = & \frac{mz\beta_+}{a(z + \beta)^3(1 + z\beta)^4} [8z^3\beta(2\beta^2 + \eta) + \beta^2(1 + z^4)(7\eta - 6) \\ & + 8z\beta(2 + \beta^2\eta) + z^2\{6 - 2\beta^4(\eta - 3) - 2\eta + \beta^2(32 + 6\eta)\}]. \end{aligned} \quad (75)$$

Applying the contour integral (64) the coefficients \mathcal{Q}_n can be computed as follows. $\beta < 1$ therefore the only singularity of \mathbb{Q} inside C_0 is at $-\beta$. The other pole at $-1/\beta$ lies outside the unit circle.

⁶Only the sign convention of h is different.

\mathbb{Q} can be expanded around its pole at $-\beta$ and keeping the coefficient of the $(z + \beta)^{-1}$ part, the result is

$$\mathcal{Q}_{-1} = 2ma^{-1}\beta_-^{-4}\beta_+ (-\beta - 8\beta^3 - 3\beta^5 + 3\beta\eta + 17\beta^3\eta + \beta^5\eta) . \quad (76)$$

In the same way with \mathbb{Q}/z one finds

$$\mathcal{Q}_0 = -2ma^{-1}\beta_-^{-4}\beta_+ (-3 - 8\beta^2 - \beta^4 + \eta + 17\beta^2\eta + 3\beta^4\eta) . \quad (77)$$

Both of the coefficients \mathcal{Q}_{-1} and \mathcal{Q}_0 are real. The differential equation for the semimajor axis is proportional to their imaginary part (57), therefore $a' = 0$. The remaining differential equations can be calculated in the same way.

The resulting nontrivial differential equations in modified time for the orbital parameters are as follows[18]. The equation for the argument of the pericenter

$$\psi' = \frac{6m}{a} \left(\frac{\beta_+}{\beta_-} \right)^2 , \quad (78)$$

and for the modified time at apocenter

$$s'_p = -\frac{m}{2a\beta_-} (\eta - 9 + \beta^2 (8\eta - 15)) . \quad (79)$$

Using the transformation to the modified time (19) in the leading order

$$\frac{d}{dt} = \frac{1}{2a} \sqrt{\frac{m}{a}} \frac{d}{ds} ,$$

from (78) it follows that

$$\dot{\psi} = \frac{3m^{3/2}}{a^{5/2} (1 - e^2)} , \quad (80)$$

which is the known expression for the precession of the pericenter[20]. The remaining differential equations have zero on the right hand side of the equation and the corresponding orbital element is constant.

4.3 Gravitational radiation reaction

Gravitational radiation damping has been recognized as a process with very important observable consequences: the PSR 1913+16 system has given the first evidence that gravitational waves exist[21], and other systems are of high importance as well [22, 23]. The equation of motion is given by [24]

$$\mathbf{a}_{RR} = -\frac{8\eta m^2}{5r^3} (-A_{5/2}\dot{r}\hat{\mathbf{n}} + B_{5/2}\mathbf{v}) , \quad (81)$$

where the subscript RR indicates the radiation reaction term and ⁷

$$\begin{aligned} A_{5/2} &= 3(1 + \rho)v^2 + \frac{1}{3}(23 + 6\gamma - 9\rho)\frac{m}{r} - 5\rho\dot{r}^2 \\ B_{5/2} &= (2 + \gamma)v^2 + (2 - \gamma)\frac{m}{r} - 3(1 + \gamma)\dot{r}^2. \end{aligned} \quad (82)$$

⁷ Our notation is slightly different from [24] as the α and β parameters are occupied; instead we use γ and ρ respectively.

The γ and ρ parameters in (82) represent the residue of gauge freedom that has not been fixed by the energy balance method and that has no physical meaning. It is known that these arbitrariness is equivalent with a coordinate transformation whose effect on the two-body separation vector is

$$\mathbf{r} \rightarrow \mathbf{r} + \delta\mathbf{r} = \mathbf{r} + \frac{8\eta m^2}{15r^2}[\rho\dot{\mathbf{r}} + (2\rho - 3\gamma)r\mathbf{v}]. \quad (83)$$

We use transformation (83) to *remove* the gauge dependency from the quaternion equation (20), after substituting (81) as the perturbing force.

In order to apply transformation (83) on the quaternion equation of motion (20) we have to rewrite it in quaternion form using modified time (19). The definition of the modified time (19) contains the separation r therefore any gauge dependent transformation of the separation, like (83), leads to gauge dependent modified time $s(\gamma, \rho)$. Consequently the transformation of any real time derivative involves a new gauge dependent contribution

$$\frac{d}{dt} = \sqrt{\frac{m}{a}} \frac{1}{2r} \frac{d}{ds} \rightarrow \sqrt{\frac{m}{a}} \frac{1}{2r} \frac{d}{ds} - \sqrt{\frac{m}{a}} \frac{1}{4r^2} \delta r \frac{d}{ds} + \mathcal{O}(\delta r^2). \quad (84)$$

It follows that transformation (83) in its original form does not cancel these new gauge dependent contributions and needs to be reparametrized. The reparametrized transformation in quaternion form is the following

$$\mathbb{U}_K \rightarrow \mathbb{U}_K + \frac{2\eta m^{5/2}}{15r^3 a^{1/2}}[K\rho r' \mathbb{U}_K + (L\rho - N\gamma)r \mathbb{U}'_K], \quad (85)$$

where K , L and N are unknown coefficients.

To obtain them consider that \mathbb{Q} (54) is Laurent series in z and gauge independence requires that every ρ and γ proportional term in the coefficients must vanish. E.g. the coefficients of $z^2\gamma$, $z^2\rho$ and $z^3\beta^2\rho$ of \mathbb{Q} (54) after simplification lead to a linear equation system which determines that $N = 3$ and $K = L = 1$ [18]. In the leading order according to (15) this is equivalent with the following real time vectorial transformation

$$\mathbf{r} \rightarrow \mathbf{r} + \frac{8\eta m^2}{15r^2}[\rho\dot{\mathbf{r}} + \frac{1}{2}(\rho - 3\gamma)r\mathbf{v}], \quad (86)$$

which is slightly different from (83).

The result from formula (57)-(63) for the semimajor axis is now gauge independent[18]

$$a' = -\frac{64m^{5/2}\eta\beta_+^3\beta_-^{-7}}{15a^{3/2}}(6 + 97\beta^2 + 219\beta^4 + 97\beta^6 + 6\beta^8), \quad (87)$$

and also for the modified eccentricity

$$\beta' = -\frac{8m^{5/2}\eta\beta_+^4\beta_-^{-6}}{15a^{5/2}}(76 + 273\beta^2 + 76\beta^4). \quad (88)$$

The remaining differential equations are trivial, with a zero on the right hand side, and the corresponding orbital elements remain constant. The gauge independent value of parameter \mathbb{D} is given in [18], while \mathbb{S} is zero.

Now we are in the position that the latter result for the semi major axis (87) and also the expression for the eccentricity (88) can be easily verified. They must be equal with the two corresponding classical formula derived from the well known Peters-Mathews formula [7], which describes the effect of gravitational radiation. After substituting the expression $e = 2\beta\beta_+^{-1}$ into (89) and (90) together with the transformation rule between the real and modified time (19) one can derive the two equation below

$$\frac{da}{dt} = -\frac{64}{5} \frac{\eta m^3}{a^3} \frac{1}{(1-e^2)^{7/2}} \left(1 + \frac{73}{24}e^2 + \frac{37}{96}e^4 \right), \quad (89)$$

$$\frac{de}{dt} = -\frac{304}{15} \frac{\eta m^3}{a^4} \frac{e}{(1-e^2)^{5/2}} \left(1 + \frac{121}{304}e^2 \right), \quad (90)$$

which are indeed identical with the two formula derived from the Peters-Mathews equation [25].

5 Conclusion and outlook

In this paper general relativity perturbations were examined using a new approach where the regularization of the Kepler problem is given with quaternions. This approach is based on the usual Kustaanheimo-Stiefel method which is defined with matrices.

With the new calculus the differential equations of the orbital parameters were derived in case when the perturbation is the leading (post)¹-Newtonian order correction of general relativity. To test the new method the precession of the pericentre is rederived.

Then the gravitational radiation reaction was analyzed, where the famous Peters-Mathews formula was reproved using the quaternion approach [7]. We have studied the gauge dependence of the equations of motion and we managed to remove the residual gauge freedom from the quaternionic equation of motion.

The new quaternionic approach is easy to implement with program code. Quaternions can be represented with pairs of complex numbers, then the equations can be calculated and solved with the help of complex analysis. This feature makes this method to a very efficient calculus for symbolic computations.

With the quaternion based regularization the spin-orbit and spin-spin interactions can be examined as well[26]. It is foreseen that these spin interaction related calculations provide the next step of our studies.

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A The components of \mathbb{U} expressed with orbital elements

These formulae can be straightforwardly derived from the unperturbed solution (28) using the expression of the rotation (32) with the rotation angles

$$\begin{aligned} U &= a^{1/2} \beta_+^{-1/2} \cos \frac{\theta}{2} \left\{ \cos \left(\omega_+ + \frac{\omega}{2} \right) + \beta \cos \left(\omega_+ - \frac{\omega}{2} \right) \right\}, \\ U_3 &= a^{1/2} \beta_+^{-1/2} \cos \frac{\theta}{2} \left\{ \sin \left(\omega_+ + \frac{\omega}{2} \right) + \beta \sin \left(\omega_+ - \frac{\omega}{2} \right) \right\}, \\ U_2 &= -a^{1/2} \beta_+^{-1/2} \sin \frac{\theta}{2} \left\{ \sin \left(\omega_- + \frac{\omega}{2} \right) + \beta \sin \left(\omega_- - \frac{\omega}{2} \right) \right\}, \\ U_1 &= a^{1/2} \beta_+^{-1/2} \sin \frac{\theta}{2} \left\{ \cos \left(\omega_- + \frac{\omega}{2} \right) + \beta \cos \left(\omega_- - \frac{\omega}{2} \right) \right\}. \end{aligned} \quad (91)$$

where $\omega_{\pm} = (\phi \pm \psi)/2$.

B Decoupling the equation of motion

For convenience the quaternion equation of motion (20) can be decoupled into two complex equations. Premultiplying (20) with $(-1 - \beta^2) \bar{\mathbb{U}}_K / (2a)$ and also postmultiplying it by \mathbb{R} while keeping only the 1, \mathbf{i} part in $\mathcal{O}(\varepsilon)$ one obtains[4]

$$\begin{aligned} & -\mathbf{i}(\beta_- - \beta z_-) \frac{a'}{2a} + \mathbf{i} z_+ \beta' + 4\mathbf{i} \beta \beta_+^{-1} \beta' \\ & + (2\beta_- + \beta z_-) Z_3 - 4\beta_+ s'_p + (1 + \beta z) \left(\mathbb{D} + 8z \frac{d\mathbb{D}}{dz} + 4z^2 \frac{d^2 \mathbb{D}}{dz^2} \right) \\ & + (1 + \beta z^{-1}) \mathbb{D}^* + (1 - \beta z) \left(\mathbb{D} + 2z \frac{d\mathbb{D}}{dz} \right) + (1 - \beta z^{-1}) \left(\mathbb{D} + 2z \frac{d\mathbb{D}}{dz} \right)^* = \\ & - (1 + \beta z^{-1}) (1 + \beta z)^2 \mathbb{Q}, \end{aligned} \quad (92)$$

where $z_{\pm} = z \pm z^{-1}$ and Z_n are the components of the angular velocity vector (12)

Similarly premultiplying equation (20) with $(1 + \beta^2) \bar{\mathbb{U}}_K \mathbf{k} / a$ and then keeping only the complex part in $\mathcal{O}(\varepsilon)$ the second complex equation is the following

[4]

$$\begin{aligned}
& -8 \frac{\beta_+ \mathbb{S}}{\beta_+ + \beta z_+} - 8 \frac{z_- \beta}{\beta_+ + \beta z_+} z \frac{d\mathbb{S}}{dz} + 8z \frac{d\mathbb{S}}{dz} + 8z^2 \frac{d^2 \mathbb{S}}{dz^2} \\
& + Z_1 \mathbf{i} \beta_- \frac{2\beta_+ z_+ + \beta (z^2 + z^{-2} + 6)}{\beta_+ + \beta z_+} - 8 \mathbf{i} \frac{\beta_+ b}{\beta_+ + \beta z_+} \\
& + Z_2 z_- \frac{\beta \beta_+ z_+ + 2(1 + \beta^4)}{\beta_+ + \beta z_+} = - (1 + \beta z^{-1}) (1 + \beta z) \mathbb{W}(z). \quad (93)
\end{aligned}$$

C The solution for \mathbb{D} and \mathbb{S}

Similarly by pairing the powers of z in the complex equations (92) and (93) two additional equation for \mathbb{D} and \mathbb{S} can be gained

$$\begin{aligned}
\mathbb{D} &= -\frac{1}{4} \sum_{\substack{n=-\infty \\ n \neq -1, 0}}^{n=\infty} \left[\frac{\beta (n + \frac{1}{2}) Q_{n-1} + (n - \frac{1}{2}) Q_n + \frac{1}{2} \overline{Q_{-n}}}{n^2 (n+1)} \right. \\
&\quad \left. + \frac{\beta^2 (n + \frac{3}{2}) Q_n + (n + \frac{1}{2}) Q_{n+1} - \frac{1}{2} \beta^2 \overline{Q_{-n-2}} - \frac{1}{2} \beta \overline{Q_{-n-1}}}{n (n+1)^2} - \frac{\frac{1}{2} \beta \overline{Q_{-n-1}}}{n^2 (n+1)^2} \right] z^n, \quad (94) \\
\mathbb{S} &= -\frac{\mathbf{i}}{4} Im \left[\sum_{n=2}^{\infty} \left(\frac{\beta \mathcal{W}_{n-1}}{(n-1)n} + \frac{\beta_+ \mathcal{W}_n}{n^2 - 1} + \frac{\beta \mathcal{W}_{n+1}}{n(n+1)} \right) z^n \right]. \quad (95)
\end{aligned}$$

D The \mathbb{D} and \mathbb{S} quantity in case of the (post)¹-newtonian effect

The complicated quantity \mathbb{D} is given only up to second β order

$$\mathbb{D} = \frac{m\beta z}{2a} (1 - 2\eta) + \frac{m\beta^2}{8az^2} (30 - 2z^4 - 9\eta + 5z^4\eta) + O(\beta^3) \quad (96)$$

while \mathbb{S} is zero.

E Gravitational radiation: \mathbb{D} and \mathbb{S}

The fairly complicated \mathbb{D} quantity is given in second β order

$$\mathbb{D} = -\frac{16}{15} \mathbf{i} z \eta \beta \left(\frac{m}{a} \right)^{5/2} + \mathbf{i} \frac{\eta \beta^2}{45z^2} \left(\frac{m}{a} \right)^{5/2} (537 + 233z^4) + O(\beta^3) \quad (97)$$

while \mathbb{S} is zero.

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